

The Ising limit of the XXZ Heisenberg magnet and certain thermal correlation functions

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In Memorium of A. N. Vassiliev

Abstract

The spin- $1/2$ XXZ Heisenberg magnet is considered for the case of the anisotropy parameter tending to infinity (so-called, *Ising limit*). A thermal correlation function of the ferromagnetic string is calculated over the ground state. The approach to the calculation of the correlation functions is based on the observation that the wave function in the limit considered is expressed in terms of the symmetric Schur functions. It is demonstrated that at low temperatures the amplitude of the asymptotical expression of the present correlation function is proportional to the squared numbers of strict boxed plane partitions.

Keywords: Heisenberg magnet, Ising limit, correlation functions

1 Introduction

The Ising model was one of the favorite models of A. N. Vassiliev. His studies were concentrated on the scaling properties of this model [1, 2]. It is remarkable that in a special limit of the anisotropy parameter tending to infinity, the Hamiltonian of the spin- $1/2$ Heisenberg XXZ chain is transformed into the Hamiltonian of one-dimensional Ising model [3]. This limit is usually called as the *Ising limit* of the XXZ chain [4]. In our paper we shall investigate the Ising limit of the XXZ magnet using an effective Hamiltonian which is commutative with the Ising Hamiltonian and has a complete system of the eigenfunctions which is common with that of the Ising Hamiltonian. This effective Hamiltonian is an appropriate object for studying of the Ising limit since it is related to the transfer-matrix of the four-vertex model, which arises in the limit of infinite anisotropy from the transfer-matrix of the six-vertex model. In their turn, the trace identities relate the XXZ Hamiltonian and the transfer-matrix of the six-vertex model.

The problem of calculation of the correlation functions of XXZ Heisenberg model in the framework of the Quantum Inverse Scattering Method [5] has required serious efforts [6–11]. We shall show that calculation of the correlation functions in the Ising limit is closely related to the theory of the symmetric functions [12], and thus is closely connected with the problems of enumerative combinatorics, such as the random walks of vicious walkers [13–17] and enumeration of boxed plane partitions [18, 19]. In this paper we shall calculate the thermal correlation functions of the ferromagnetic string over the ground state using the approach developed in [20–24].

The XXZ spin- $1/2$ Heisenberg magnet is defined on the one-dimensional lattice consisting of $M + 1$ sites labeled by elements of the set $\mathcal{M} \equiv \{0 \leq k \leq M, k \in \mathbb{Z}\}$, $M + 1 = 0 \pmod{2}$. The corresponding spin Hamiltonian is defined as follows:

$$\hat{H}_{XXZ} = -\frac{1}{2} \sum_{k=0}^M (\sigma_{k+1}^- \sigma_k^+ + \sigma_{k+1}^+ \sigma_k^- + \frac{\Delta}{2} (\sigma_{k+1}^z \sigma_k^z - 1) + h \sigma_k^z), \quad (1)$$

where the parameter $\Delta \in \mathbb{R}$ is the anisotropy of the model, and h is the external magnetic field. The local spin operators $\sigma_k^\pm = \frac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$ and σ_k^z , dependent on the lattice argument $k \in \mathcal{M}$, are defined standardly (see [6, 7]), and their commutation rules are given by the relations: $[\sigma_k^+, \sigma_l^-] = \delta_{k,l} \sigma_l^z$, $[\sigma_k^z, \sigma_l^\pm] = \pm 2\delta_{k,l} \sigma_l^\pm$. The state-space is spanned over of the state-vectors $\bigotimes_{k=0}^M |s\rangle_k$, where s implies either \uparrow or \downarrow . The spin “up” and “down” states ($|\uparrow\rangle$ and $|\downarrow\rangle$, respectively) provide a natural basis in the linear space \mathbb{C}^2 so that

$$|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The periodic boundary conditions $\sigma_{k+(M+1)}^\# = \sigma_k^\#$ are imposed. The Hamiltonian (1) commutes with the operator of third component of the total spin \hat{S}^z :

$$[\hat{H}_{XXZ}, \hat{S}^z] = 0, \quad \hat{S}^z \equiv \frac{1}{2} \sum_{k=0}^M \sigma_k^z. \quad (2)$$

To represent N -particle state-vectors, $|\Psi_N(u_1, \dots, u_N)\rangle$, let the sites with spin “down” states be labeled by the coordinates μ_i forming a strict partition $\boldsymbol{\mu} \equiv (\mu_1, \mu_2, \dots, \mu_N)$,

where $M \geq \mu_1 > \mu_2 > \dots > \mu_N \geq 0$. There is a correspondence between each partition and an appropriate sequence of zeros and unities of the form: $\{e_k \equiv e_k(\boldsymbol{\mu})\}_{k \in \mathcal{M}}$, where $e_k \equiv \delta_{k, \mu_n}$, $1 \leq n \leq N$. The Hamiltonian (1) is diagonalised *via* the following ansatz:

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\{e_k(\boldsymbol{\mu})\}_{k \in \mathcal{M}}} \chi_{\boldsymbol{\mu}}^{\text{XXZ}}(\mathbf{u}) \prod_{k=0}^M (\sigma_k^-)^{e_k} |\uparrow\rangle, \quad (3)$$

where the sum is taken over C_M^N strict partitions $\boldsymbol{\mu}$, and $|\uparrow\rangle \equiv \bigotimes_{n=0}^M |\uparrow\rangle_n$ is the fully polarized state with all spins “up”. It is proposed to use bold-faced letters as short-hand notations for appropriate N -tuples of numbers: for instance, \mathbf{u} instead of (u_1, \dots, u_N) , etc. Therefore, the wave function $\chi_{\boldsymbol{\mu}}^{\text{XXZ}}(\mathbf{u})$ in (3) is of the form:

$$\chi_{\boldsymbol{\mu}}^{\text{XXZ}}(\mathbf{u}) = \sum_{S_{p_1, p_2, \dots, p_N}} \mathcal{A}_S(\mathbf{u}) u_{p_1}^{2\mu_1} u_{p_2}^{2\mu_2} \dots u_{p_N}^{2\mu_N}, \quad (4)$$

$$\mathcal{A}_S(\mathbf{u}) \equiv \prod_{1 \leq j < i \leq N} \frac{1 - 2\Delta u_{p_i}^2 + u_{p_i}^2 u_{p_j}^2}{u_{p_i}^2 - u_{p_j}^2}, \quad (5)$$

where summation goes over all elements of the symmetric group

$$S_{p_1, p_2, \dots, p_N} \equiv S \begin{pmatrix} 1, & 2, & \dots, & N \\ p_1, & p_2, & \dots, & p_N \end{pmatrix}.$$

The state-vectors (3) are the eigen-states of the Hamiltonian (1), if and only if the variables u_l satisfy the *Bethe equations*

$$u_l^{2(M+1)} = (-1)^{N-1} \prod_{k=1}^N \frac{1 - 2\Delta u_l^2 + u_l^2 u_k^2}{1 - 2\Delta u_k^2 + u_l^2 u_k^2}, \quad 1 \leq l \leq N. \quad (6)$$

The corresponding eigen-energies are given by

$$E_N(\mathbf{u}) = -\frac{1}{2} \sum_{i=1}^N (u_i^2 + u_i^{-2} - 2\Delta - 2h). \quad (7)$$

The limit of zero anisotropy, $\Delta \rightarrow 0$, transforms (1) to the Hamiltonian of the XX model on the periodic chain

$$\hat{H}_{\text{XX}} \equiv -\frac{1}{2} \sum_{k=0}^M (\hat{h}_{k+1, k} + h \sigma_k^z), \quad \hat{h}_{k+1, k} \equiv \sigma_{k+1}^- \sigma_k^+ + \sigma_{k+1}^+ \sigma_k^-. \quad (8)$$

This limit is known as the free fermion limit of the XXZ chain, and it is the most studied one [25–28].

In the next section II we shall consider the Ising limit corresponding to the limit $\Delta \rightarrow -\infty$. The thermal correlation functions of the ferromagnetic string operator are obtained in Section III. Section IV presents several results for the four-vertex model which is the $\Delta \rightarrow -\infty$ limit of the six-vertex model. In Section IV it is demonstrated that the effective Hamiltonian commutes with the transfer-matrix of the four-vertex model, the orthogonality of the eigen-functions is proved, and the relation between the four-vertex model and the boxed plane partitions [12] is discussed. The low temperature asymptotics of the correlation function is studied in the Section V. Discussion in Section VI concludes the paper.

2 The Ising limit of the XXZ chain

Less studied limit of the XXZ model is the *Strong Anisotropy* (SA) limit $\Delta \rightarrow -\infty$ [4, 29–31]. This limit is called the *Ising limit* since the simplest Hamiltonian in this limit turns out to be the Hamiltonian of the one-dimensional Ising model [4]:

$$\lim_{\Delta \rightarrow -\infty} \frac{1}{\Delta} \hat{H}_{XXZ} = \hat{H}_{\text{Is}} \equiv \frac{-1}{4} \sum_{k=0}^M (\sigma_{k+1}^z \sigma_k^z - 1). \quad (9)$$

However we shall study the strong anisotropy limit using the effective Hamiltonian \hat{H}_{SA} , which is formally equivalent to the XX Hamiltonian (8), provided the latter is supplied with the requirement forbidding two spin “down” states to occupy any pair of nearest-neighbor sites:

$$\hat{H}_{\text{SA}} = -\frac{1}{2} \sum_{k=0}^M \mathcal{P} (\hat{h}_{k+1,k} + h \sigma_k^z) \mathcal{P}, \quad \mathcal{P} \equiv \prod_{k=0}^M (\mathbb{I} - \hat{q}_{k+1} \hat{q}_k). \quad (10)$$

The projectors \mathcal{P} “cut out” the spin “down” states for any pair of nearest-neighbor sites. The local projectors onto the spin “up” and “down” states are of the form:

$$\hat{q}_k \equiv \frac{1}{2} (\mathbb{I} - \sigma_k^z), \quad \check{q}_k \equiv \frac{1}{2} (\mathbb{I} + \sigma_k^z), \quad \check{q}_k + \hat{q}_k = \mathbb{I}, \quad k \in \mathcal{M}, \quad (11)$$

where \mathbb{I} is the unit operator in $(M+1)$ -fold tensor product of the linear spaces \mathbb{C} . The point is that the Hamiltonians (9) and (10) are commutative. Indeed, the Hamiltonian (9) can be represented by means of (11) as follows:

$$\hat{H}_{\text{Is}} = \hat{N} - \sum_{k=0}^M \hat{q}_{k+1} \hat{q}_k, \quad \hat{N} \equiv \sum_{k=0}^M \hat{q}_k, \quad (12)$$

where \hat{N} is the operator of number of particles commuting with the Hamiltonians \hat{H}_{XX} and \hat{H}_{SA} . On another hand, the definition of the projector \mathcal{P} yields:

$$\mathcal{P} \sum_{k=0}^M \hat{q}_{k+1} \hat{q}_k = \sum_{k=0}^M \hat{q}_{k+1} \hat{q}_k \mathcal{P} = 0.$$

Then, $[\hat{H}_{\text{SA}}, \hat{H}_{\text{Is}}] = 0$ and so the Hamiltonians (9) and (10) possess a common system of the eigen-functions.

The wave function (4) takes the form in the limit $\Delta \rightarrow -\infty$:

$$\chi_{\boldsymbol{\mu}}^{\text{SA}}(\mathbf{u}) = \det(u_j^{2(\mu_k - N + k)})_{1 \leq j, k \leq N} \prod_{1 \leq n < l \leq N} (u_l^2 - u_n^2)^{-1}, \quad (13)$$

where the coordinates of the spin “down” states form a strict decreasing partition $\boldsymbol{\mu}$, as in (3). It follows from (13) that the wave function is not equal to zero if and only if the elements μ_i , $1 \leq i \leq N$, satisfy the *exclusion condition*: $\mu_i > \mu_{i+1} + 1$. In other words, in the considered limit occupation of nearest sites is forbidden, and the *hard-core diameter*

equal to duplicated inter-site separation arises. The exponential parametrization $u_j^2 = e^{i\theta_j}$ brings the Bethe equations (6) to the form:

$$e^{i(M+1-N)\theta_k} = (-1)^{N-1} \prod_{j=1}^N e^{-i\theta_j}, \quad 1 \leq k \leq N, \quad (14)$$

which is respected by the solutions

$$\theta_j = \frac{2\pi}{M+1-N} \left(I_j - \frac{N-1}{2} - P \right), \quad (15)$$

where $P \equiv \frac{1}{2\pi} \sum_{j=1}^N \theta_j$, and the I_j are integers or half-integers, depending on whether N is odd or even. It suffices to consider a set of N different numbers I_j satisfying the condition $M - N \geq I_1 > I_2 > \dots > I_N \geq 0$. The ground state of the model is defined by the solutions

$$\theta_j^v \equiv \frac{2\pi}{M+1-N} \left(N - j - \frac{N-1}{2} \right), \quad (16)$$

with the condition $P = 0$ fulfilled. The notation $\boldsymbol{\theta}$ for N -tuple $(\theta_1, \theta_2, \dots, \theta_N)$ of the solutions (15) will be especially convenient further in order to stress that one is concerned with the solution of the Bethe equations. Otherwise, it is appropriate to use \mathbf{u} as an indication that an arbitrary parametrization of the wave-function (13) is meant. The eigen-energy of the ground state of the model calculated by means of (7) and (16) is:

$$E_N(\boldsymbol{\theta}^v) = hN - \operatorname{cosec} \frac{\pi}{M+1-N} \sin \frac{\pi N}{M+1-N}. \quad (17)$$

The eigen-energy of the Ising Hamiltonian (9) has the form: $E_{\text{Is}} = N$.

The two limits, (8) and (10), are similar in the sense that their wave functions are expressible through the *Schur functions* [12]:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}) \equiv S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_N) \equiv \frac{\det(x_j^{\lambda_k + N - k})_{1 \leq j, k \leq N}}{\det(x_j^{N - k})_{1 \leq j, k \leq N}}, \quad (18)$$

where $\boldsymbol{\lambda}$ denotes the partition $(\lambda_1, \lambda_2, \dots, \lambda_N)$, which is an N -tuple of nonincreasing non-negative integers: $L \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$. Indeed, any strict partition $M \geq \mu_1 > \mu_2 > \dots > \mu_N \geq 0$ and non-strict partition $M+1-N \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$ (denoted as $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$, respectively) can be related by the formula $\lambda_j = \mu_j - N + j$, where $1 \leq j \leq N$. In other terms, $\boldsymbol{\lambda} = \boldsymbol{\mu} - \boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is the strict partition $(N-1, N-2, \dots, 1, 0)$. Any strict partition $\boldsymbol{\mu}$ with the elements respecting the exclusion condition $\mu_i > \mu_{i+1} + 1$ is connected with the non-strict partition $\tilde{\boldsymbol{\lambda}}$ by the relation $\tilde{\boldsymbol{\lambda}} = \boldsymbol{\mu} - 2\boldsymbol{\delta}$, where $M+2(1-N) \geq \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \dots \geq \tilde{\lambda}_N \geq 0$. So, the wave function (13) may be represented as $\chi_{\boldsymbol{\mu}}^{\text{SA}}(\mathbf{u}) = S_{\tilde{\boldsymbol{\lambda}}}(\mathbf{u}^2)$, and the corresponding state-vector (3) is specialized as follows:

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\tilde{\boldsymbol{\lambda}} \subseteq \{(M-2(N-1))^N\}} S_{\tilde{\boldsymbol{\lambda}}}(\mathbf{u}^2) \prod_{k=0}^M (\sigma_k^-)^{e_k} |\uparrow\rangle. \quad (19)$$

Summation in (19) goes over all non-strict partitions $\tilde{\lambda}$ into at most N parts so that each is less than $M - 2(N - 1)$, i.e., $M - 2(N - 1) \geq \tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \dots \geq \tilde{\lambda}_N \geq 0$. Summation over the non-strict partitions $\tilde{\lambda}$ is equivalent to that over the strict partitions μ with the elements respecting the conditions $\mu_i > \mu_{i+1} + 1$ and $\tilde{\lambda} = \mu - 2\delta$.

3 Survival probability of the ferromagnetic string

We shall study the thermal correlation function of the states with no spins “down” on the last $n + 1$ sites of the lattice (would be called as *survival probability of the ferromagnetic string*) defined by the ratio:

$$\mathcal{T}(\theta^v, n, \beta) \equiv \frac{\langle \Psi_N(\theta^v) | \bar{\Pi}_n e^{-\beta \hat{H}_{\text{SA}}} \bar{\Pi}_n | \Psi_N(\theta^v) \rangle}{\langle \Psi_N(\theta^v) | e^{-\beta \hat{H}_{\text{SA}}} | \Psi_N(\theta^v) \rangle}, \quad \bar{\Pi}_n \equiv \prod_{j=M-n}^M \check{q}_j, \quad (20)$$

where \hat{H}_{SA} and $\bar{\Pi}_n$ are defined, respectively, by (10) and (11), and β is inverse temperature. Besides, θ^v indicates that the eigen-states are calculated on the solution (16). Our calculations will be based on the following relation for the Schur functions (18) which is due to the Binet–Cauchy formula [32]:

$$\begin{aligned} \sum_{\lambda \subseteq \{L^N\}} S_{\lambda}(x_1^2, \dots, x_N^2) S_{\lambda}(y_1^2, \dots, y_N^2) &= \det \left(\frac{1 - (x_k y_j)^{2(N+L)}}{1 - (x_k y_j)^2} \right)_{1 \leq j, k \leq N} \\ &\times \prod_{1 \leq k < j \leq N} (y_j^2 - y_k^2)^{-1} \prod_{1 \leq m < l \leq N} (x_l^2 - x_m^2)^{-1}. \end{aligned} \quad (21)$$

Summation in (21) goes over all non-strict partitions λ into at most N parts so that each is less than L : $L \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$. The notation for the range of summation in (21) will be extensively used in the rest of the paper.

Firstly, it is straightforward to calculate, using (21), the following average of the projector $\bar{\Pi}_n$ (10) over the state-vectors (19):

$$\begin{aligned} \langle \Psi_N(\mathbf{v}) | \bar{\Pi}_n | \Psi_N(\mathbf{u}) \rangle &= \sum_{\tilde{\lambda} \subseteq \{(M-2N-n+1)^N\}} S_{\tilde{\lambda}}(\mathbf{v}^{-2}) S_{\tilde{\lambda}}(\mathbf{u}^2) \\ &= \frac{1}{\mathcal{V}(\mathbf{u}^2) \mathcal{V}(\mathbf{v}^{-2})} \det \left(\frac{1 - (u_k^2/v_j^2)^{M-N-n+1}}{1 - u_k^2/v_j^2} \right)_{1 \leq j, k \leq N}, \end{aligned} \quad (22)$$

where the notation for the Vandermonde determinant is used:

$$\mathcal{V}(\mathbf{u}^2) \equiv \prod_{1 \leq m < l \leq N} (u_l^2 - u_m^2). \quad (23)$$

Right-hand side of (22), taken at $n + 1 = 0$, gives the answer for the scalar product of the state-vectors (19), i.e., for $\langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle$. Equation (22), taken on the solutions (16), allows one to obtain the so-called *Emptiness Formation Probability* $E(n, N)$, which expresses the probability of formation of a string of the empty (i.e., spin “up”) states on

last $n + 1$ sites of the lattice:

$$E(n, N) \equiv \frac{\langle \Psi_N(\boldsymbol{\theta}^v) | \bar{\Pi}_n | \Psi_N(\boldsymbol{\theta}^v) \rangle}{\langle \Psi_N(\boldsymbol{\theta}^v) | \Psi_N(\boldsymbol{\theta}^v) \rangle} = \frac{M - N + 1}{M + 1} \det \left(\left(1 - \frac{n}{M - N + 1} \right) \delta_{lj} \right. \\ \left. + \frac{1 - e^{in(k_l - k_j)}}{(M - N + 1)(1 - e^{i(k_l - k_j)})} (1 - \delta_{lj}) \right)_{1 \leq l, j \leq N}, \quad k_l \equiv \frac{2\pi l}{M + 1 - N}.$$

The solutions of the Bethe equations (14) constitute a complete set of the eigen-states, and therefore one can obtain the resolution of the identity operator:

$$\mathbb{I} = \frac{1}{(M + 1)(M + 1 - N)^{N-1}} \sum_{\{\boldsymbol{\theta}\}} |\mathcal{V}(e^{i\boldsymbol{\theta}})|^2 |\Psi_N(\boldsymbol{\theta})\rangle \langle \Psi_N(\boldsymbol{\theta})|, \quad (24)$$

where the summation over $\{\boldsymbol{\theta}\}$ implies summation over all independent solutions (15). Here the exponential parametrization for the solutions of the Bethe equations is used in the compact form, $e^{i\boldsymbol{\theta}} \equiv (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N})$. We shall calculate the nominator of (20) using insertions of the resolution (24). Taking into account that

$$\langle \Psi_N(\mathbf{v}) | e^{-\beta \hat{H}_{\text{SA}}} | \Psi_N(\boldsymbol{\theta}) \rangle = \langle \Psi_N(\mathbf{v}) | \Psi_N(\boldsymbol{\theta}) \rangle e^{-\beta E_N(\boldsymbol{\theta})},$$

we obtain:

$$\langle \Psi_N(\mathbf{v}) | \bar{\Pi}_n e^{-\beta \hat{H}_{\text{SA}}} \bar{\Pi}_n | \Psi_N(\mathbf{u}) \rangle = \frac{1}{(M + 1)(M + 1 - N)^{N-1}} \sum_{\{\boldsymbol{\theta}\}} e^{-\beta E_N(\boldsymbol{\theta})} \\ \times |\mathcal{V}(e^{i\boldsymbol{\theta}})|^2 \mathcal{P}(\mathbf{v}^{-2}, e^{i\boldsymbol{\theta}}) \mathcal{P}(e^{-i\boldsymbol{\theta}}, \mathbf{u}^2), \quad (25)$$

where

$$\mathcal{P}(\mathbf{v}^{-2}, e^{i\boldsymbol{\theta}}) \equiv \sum_{\tilde{\boldsymbol{\lambda}}^L} S_{\tilde{\boldsymbol{\lambda}}^L}(\mathbf{v}^{-2}) S_{\tilde{\boldsymbol{\lambda}}^L}(e^{i\boldsymbol{\theta}}), \quad \mathcal{P}(e^{-i\boldsymbol{\theta}}, \mathbf{u}^2) \equiv \sum_{\tilde{\boldsymbol{\lambda}}^R} S_{\tilde{\boldsymbol{\lambda}}^R}(e^{-i\boldsymbol{\theta}}) S_{\tilde{\boldsymbol{\lambda}}^R}(\mathbf{u}^2). \quad (26)$$

The range of summation in (26) is taken as follows: $\tilde{\boldsymbol{\lambda}}^L, \tilde{\boldsymbol{\lambda}}^R \subseteq \{(M - 2N - n + 1)^N\}$. Then, using the relation (21) we obtain from (25):

$$\langle \Psi_N(\mathbf{v}) | \bar{\Pi}_n e^{-\beta \hat{H}_{\text{SA}}} \bar{\Pi}_n | \Psi_N(\mathbf{u}) \rangle = \\ = \frac{1}{(M + 1)(M + 1 - N)^{N-1}} \sum_{M-N \geq I_1 > I_2 \dots > I_N \geq 0} e^{-\beta E_N(\boldsymbol{\theta})} \\ \times \det \left(\frac{1 - (e^{i\theta_i} v_j^{-2})^{M-N-n+1}}{1 - e^{i\theta_i} v_j^{-2}} \right)_{1 \leq i, j \leq N} \det \left(\frac{1 - (u_l^2 e^{-i\theta_p})^{M-N-n+1}}{1 - u_l^2 e^{-i\theta_p}} \right)_{1 \leq l, p \leq N}, \quad (27)$$

where summation goes over the ordered sets $\{I_k\}_{1 \leq k \leq N}$ that parametrize $\boldsymbol{\theta}$ (15), and $E_N(\boldsymbol{\theta})$ is given by (17). Expression for $\mathcal{T}(\boldsymbol{\theta}^v, n, \beta)$ (20) appears from (27) as follows:

$$\mathcal{T}(\boldsymbol{\theta}^v, n, \beta) = \frac{1}{(M + 1)^2 (M + 1 - N)^{N-2}} \sum_{M-N \geq I_1 > I_2 \dots > I_N \geq 0} e^{-\beta(E_N(\boldsymbol{\theta}) - E_N(\boldsymbol{\theta}^v))} \\ \times \left| \det \left(\frac{1 - e^{i(M-N-n+1)(\theta_l - \theta_p^v)}}{1 - e^{i(\theta_l - \theta_p^v)}} \right)_{1 \leq p, l \leq N} \right|^2, \quad (28)$$

where $\boldsymbol{\theta}^v$ implies the ground-state solution (16), and $E_N(\boldsymbol{\theta}^v)$ is the ground-state energy in the limit of strong anisotropy.

4 Four-vertex model and boxed plane partitions

4.1 The XXZ Hamiltonian at $\Delta \rightarrow -\infty$ and the four-vertex model

The *six-vertex* model on a square lattice is defined by six different configurations of arrows pointed both in and out of each lattice site. A statistical weight w_k ($k = 1, 2, \dots, 6$) is ascribed to each admissible type of the vertices (Fig. 1). Representing the arrows pointing up or to the right by the solid lines one can get the alternative description of the vertices in terms of lines floating through the lattice sites. So far the bonds of a lattice may be only in two states – with a line either without it, there is a one to one correspondence between the admissible configuration of arrows on a lattice and the network of lines – the nest of lattice paths. The *four-vertex* model is obtained in the limit when the weights $w_1 = w_3 = 0$.

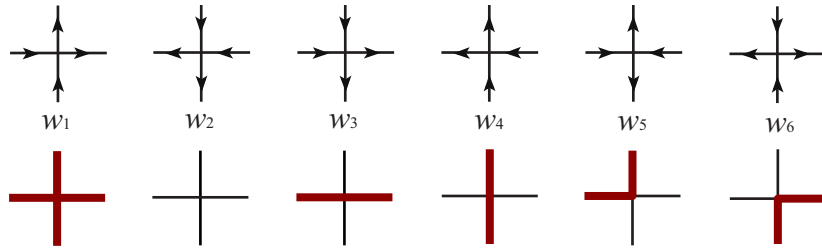


Figure 1: The vertex configurations of the 6-vertex model.

The L -operator of the six-vertex model is equal to [36]:

$$L_{6v}(n|u) = \begin{pmatrix} -ue^{\gamma\sigma_n^z} - u^{-1}e^{-\gamma\sigma_n^z} & 2\sinh(2\gamma)\sigma_n^- \\ 2\sinh(2\gamma)\sigma_n^+ & ue^{-\gamma\sigma_n^z} + u^{-1}e^{\gamma\sigma_n^z} \end{pmatrix}, \quad (29)$$

where $u \in \mathbb{C}$ and $\gamma \equiv \frac{1}{2}\text{Arch}\Delta$, and it satisfies the intertwining relation:

$$\tilde{R}(u, v)(L_{6v}(n|u) \otimes L_{6v}(n|v)) = (L_{6v}(n|v) \otimes L_{6v}(n|u))\tilde{R}(u, v). \quad (30)$$

Here, $\tilde{R}(u, v)$ is the (4×4) -matrix:

$$\tilde{R}(u, v) = \begin{pmatrix} \tilde{f}(v, u) & 0 & 0 & 0 \\ 0 & \tilde{g}(v, u) & 1 & 0 \\ 0 & 1 & \tilde{g}(v, u) & 0 \\ 0 & 0 & 0 & \tilde{f}(v, u) \end{pmatrix}, \quad (31)$$

where

$$\tilde{f}(v, u) = \frac{u^2e^{2\gamma} - v^2e^{-2\gamma}}{u^2 - v^2}, \quad \tilde{g}(v, u) = \frac{uv}{u^2 - v^2} (e^{2\gamma} - e^{-2\gamma}), \quad (32)$$

and $u, v \in \mathbb{C}$.

Let us consider the following transformation of the L -operator (29):

$$\begin{aligned} \check{L}(n|u) &= e^{h\sigma_n^z} e^{(\omega/2)\sigma_n^z} L_{6v}(n|u) e^{-(\omega/2)\sigma_n^z} \\ &= \begin{pmatrix} -ue^{(h+\gamma)\sigma_n^z} - u^{-1}e^{(h-\gamma)\sigma_n^z} & 2\sinh(2\gamma)e^{\omega+h\sigma_n^z}\sigma_n^- \\ 2\sinh(2\gamma)e^{-\omega+h\sigma_n^z}\sigma_n^+ & ue^{(h-\gamma)\sigma_n^z} + u^{-1}e^{(h+\gamma)\sigma_n^z} \end{pmatrix}. \end{aligned} \quad (33)$$

The operator (33) is intertwined by the transformed R -matrix of the form:

$$\begin{aligned}\check{R}(u, v) &= (1 \otimes e^{-h\sigma^z}) \tilde{R}(u, v) (1 \otimes e^{h\sigma^z}) \\ &= \begin{pmatrix} \tilde{f}(v, u) & 0 & 0 & 0 \\ 0 & \tilde{g}(v, u) & e^{2h} & 0 \\ 0 & e^{-2h} & \tilde{g}(v, u) & 0 \\ 0 & 0 & 0 & \tilde{f}(v, u) \end{pmatrix}.\end{aligned}\quad (34)$$

If one puts $h = \omega = \gamma$, then the limit

$$\lim_{\gamma \rightarrow \infty} e^{-2\gamma} \check{L}(n|u) = L_{4v}(n|u) \equiv \begin{pmatrix} -u\check{q}_n & \sigma_n^- \\ \sigma_n^+ & u^{-1}\check{q}_n \end{pmatrix}, \quad (35)$$

where $\check{q}_n = \sigma_n^+ \sigma_n^-$ (see (11)), gives us the L -operator of, so-called, *four-vertex* model. Analogously, the limit $\gamma \rightarrow \infty$ transforms the matrix $e^{-2\gamma} \check{R}(u, v)$ into the R -matrix of the four-vertex model [22]:

$$R(u, v) = \begin{pmatrix} f(v, u) & 0 & 0 & 0 \\ 0 & g(v, u) & 1 & 0 \\ 0 & 0 & g(v, u) & 0 \\ 0 & 0 & 0 & f(v, u) \end{pmatrix}, \quad (36)$$

where

$$f(v, u) = \frac{u^2}{u^2 - v^2}, \quad g(v, u) = \frac{uv}{u^2 - v^2}. \quad (37)$$

The monodromy matrix of the models is defined as the matrix product of L -operators:

$$T(u) = L(M|u)L(M-1|u) \cdots L(0|u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \quad (38)$$

The transfer matrix is the matrix trace of the monodromy matrix: $\tau(u) = \text{Tr } T(u)$. In [33] it was proved that the transfer-matrix of the six-vertex model commutes with the Hamiltonian of the XXZ model (1): $[\hat{H}_{6v}, \tau_{6v}(u)] = 0$.

The transfer-matrix of the four-vertex model $\tau_{4v}(u)$ satisfies the property:

$$\mathcal{P}\tau_{4v}(u) = \tau_{4v}(u)\mathcal{P} = \tau_{4v}(u), \quad (39)$$

where \mathcal{P} is the projector defined in (10). Indeed, using the explicit expression for the entries of the product of two L -operators,

$$L(n+1|u)L(n|u) = \begin{pmatrix} A_{n+1,n} & B_{n+1,n} \\ C_{n+1,n} & D_{n+1,n} \end{pmatrix}, \quad (40)$$

one can show that

$$\mathcal{C}_{n+1,n} L(n+1|u)L(n|u) = L(n+1|u)L(n|u) \mathcal{C}_{n+1,n} = L(n+1|u)L(n|u), \quad (41)$$

where $\mathcal{C}_{n+1,n} \equiv 1 - \hat{q}_{n+1}\hat{q}_n$. It follows from (41) that $\mathcal{C}_{n+1,n} \tau_{4v}(u) = \tau_{4v}(u) \mathcal{C}_{n+1,n} = \tau_{4v}(u)$ at $n = 0, \dots, M-1$. Invariance of the matrix trace with respect of cyclic permutation

implies that $\mathcal{C}_{0,M} \tau_{4v}(u) = \tau_{4v}(u) \mathcal{C}_{0,M} = \tau_{4v}(u)$, and therefore the relation (39) is also fulfilled.

Further, one can advance the following

Proposition: *The transfer matrix of the four-vertex model commutes with the XXZ Hamiltonian taken in the Ising limit:*

$$[\widehat{H}_{SA}, \tau_{4v}(u)] = 0. \quad (42)$$

Proof: In order to prove (42), it suffices to consider \widehat{H}_{SA} (10) at $h = 0$. We shall use the method proposed in [33] and then developed in [34]. Generalizing this approach with respect of the problem under consideration, we shall take into account the relation (subscripts are omitted):

$$\begin{aligned} \mathcal{C}_{n+1,n} [\widehat{h}_{n+1,n}, L(n+1|u)L(n|u)] \mathcal{C}_{n+1,n} \\ = \mathcal{C}_{n+1,n} (Q(n+1|u)L(n|u) - L(n+1|u)Q(n|u)) \mathcal{C}_{n+1,n}, \end{aligned} \quad (43)$$

where the operators $\widehat{h}_{n+1,n}$ and $\mathcal{C}_{n+1,n}$ are introduced in (8) and (41), respectively. If so, we obtain the vanishing of the commutator in question,

$$[\widehat{H}, \tau(u)] = -\frac{1}{2} \sum_{n=0}^M \mathcal{P} [\widehat{h}_{n+1,n}, \tau(u)] \mathcal{P} = -\frac{1}{2} \sum_{n=0}^M \text{Tr}(\mathcal{P} [\widehat{h}_{n+1,n}, T(u)] \mathcal{P}) = 0, \quad (44)$$

provided that

$$[\widehat{h}_{n+1,n}, T(u)] = L(M|u) \cdots L(n+2|u) [\widehat{h}_{n+1,n}, L(n+1|u)L(n|u)] L(n-1|u) \cdots L(0|u). \quad (45)$$

Furthermore, the commutators of the entries from the right-hand side of (40) look as follows:

$$\begin{aligned} [\widehat{h}_{n+1,n}, A_{n+1,n}] &= -[\widehat{h}_{n+1,n}, D_{n+1,n}] = \check{q}_{n+1} - \check{q}_n \\ [\widehat{h}_{n+1,n}, B_{n+1,n}] &= -u\sigma_{n+1}^- + u^{-1}\sigma_n^- + u\sigma_{n+1}^-\hat{q}_n - u^{-1}\hat{q}_{n+1}\sigma_n^-, \\ [\widehat{h}_{n+1,n}, C_{n+1,n}] &= -u^{-1}\sigma_{n+1}^+ + u\sigma_n^+ + u^{-1}\sigma_{n+1}^+\hat{q}_n - u\hat{q}_{n+1}\sigma_n^+. \end{aligned} \quad (46)$$

In turn, one obtains from (46):

$$\begin{aligned} \mathcal{C}_{n+1,n} [\widehat{h}_{n+1,n}, A_{n+1,n}] \mathcal{C}_{n+1,n} &= \mathcal{C}_{n+1,n} (\check{q}_{n+1} - \check{q}_n) \mathcal{C}_{n+1,n}, \\ \mathcal{C}_{n+1,n} [\widehat{h}_{n+1,n}, B_{n+1,n}] \mathcal{C}_{n+1,n} &= \mathcal{C}_{n+1,n} (-u\sigma_{n+1}^- + u^{-1}\sigma_n^-) \mathcal{C}_{n+1,n}, \\ \mathcal{C}_{n+1,n} [\widehat{h}_{n+1,n}, C_{n+1,n}] \mathcal{C}_{n+1,n} &= \mathcal{C}_{n+1,n} (-u^{-1}\sigma_{n+1}^+ + u\sigma_n^+) \mathcal{C}_{n+1,n}, \\ \mathcal{C}_{n+1,n} [\widehat{h}_{n+1,n}, D_{n+1,n}] \mathcal{C}_{n+1,n} &= \mathcal{C}_{n+1,n} (-\check{q}_{n+1} + \check{q}_n) \mathcal{C}_{n+1,n}. \end{aligned} \quad (47)$$

On the other hand, using the diagonal matrix

$$Q(n|u) = \begin{pmatrix} u^{-1}\sigma^0 & 0 \\ 0 & u\sigma^0 \end{pmatrix},$$

we obtain the following relation for the L -operators (35):

$$Q(n+1|u)L(n|u) - L(n+1|u)Q(n|u) = \begin{pmatrix} \check{q}_{n+1} - \check{q}_n & -u\sigma_{n+1}^- + u^{-1}\sigma_n^- \\ -u^{-1}\sigma_{n+1}^+ + u\sigma_n^+ & -\check{q}_{n+1} + \check{q}_n \end{pmatrix}. \quad (48)$$

Comparison of (47) and (48) demonstrates that the relation (43), as well as the statement of the relation (42) are valid indeed. ■

The state vector of the four-vertex model is constructed in the framework of the algebraic Bethe ansatz as follows:

$$|\Psi_N(\mathbf{u})\rangle = \prod_{i=1}^N B_{4v}(u_i) |\uparrow\rangle. \quad (49)$$

This vector is the eigen-vector both of the transfer-matrix of the four-vertex model and of the Hamiltonian (10), provided that the parameters u_l fulfill the Bethe equations (14). It was shown in [22] that this vector can be represented in the form (19).

According to the representation (22) taken at $n + 1 = 0$, the scalar product $\langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle$ is proportional to the determinant of the matrix, say, $(T_{kj})_{1 \leq k, j \leq N}$. The Bethe equations (14) enable one to represent this determinant as follows:

$$\det(T_{kj})_{1 \leq k, j \leq N} \equiv \det\left(\frac{1 - (u_k^2/v_j^2)^{M-N+2}}{1 - u_k^2/v_j^2}\right) = \left(\prod_{k=1}^N \frac{-v_k}{u_k}\right)^N \det(A_{kj})_{1 \leq k, j \leq N}, \quad (50)$$

where the entries A_{kj} take the form:

$$A_{kj} = \frac{(u_k v_j)^{M+3-N}}{u_k^2 - v_j^2} \left(\prod_{l=1, l \neq k}^N u_l^2 - \prod_{l=1, l \neq j}^N v_l^2 \right). \quad (51)$$

It is crucial that the determinant of the matrix $(A_{kj})_{1 \leq k, j \leq N}$ vanishes provided the parameters \mathbf{u} and \mathbf{v} are independent solutions of the Bethe equations (14). Really, this matrix has a non-trivial eigen-vector χ_j with zero eigen-value:

$$\sum_{j=1}^N A_{kj} \chi_j = 0, \quad \chi_j = (u_j^2 - v_j^2) \prod_{l=1, l \neq j}^N \frac{u_j^2 - v_l^2}{u_j^2 - u_l^2}. \quad (52)$$

Validity of (52) is justified by two identities:

$$\begin{aligned} \sum_{k=1}^N \left(\frac{y_k - x_k}{y_k - x_j} \prod_{l=1, l \neq k}^N \frac{y_k - x_l}{y_k - y_l} \right) &= 1, \\ \sum_{k=1}^N \left(\frac{y_k - x_k}{y_k} \prod_{l=1, l \neq k}^N \frac{y_k - x_l}{y_k - y_l} \right) &= 1 - \prod_{l=1}^N \frac{x_l}{y_l}. \end{aligned}$$

Therefore, the scalar product $\langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle$ vanishes (i.e., the state-vectors are orthogonal) provided the sets of the parameters \mathbf{u} and \mathbf{v} represent independent Bethe solutions. Completeness of the system of the state-vectors was proved in [4, 30].

4.2 Boxed plane partitions

Consider the four-vertex model [22]. There is one to one correspondence between the strict plane partitions (i.e., the plane partitions that decay along each column and each row [12]) and the nests of the admissible paths on a square lattice of the size $2N \times (M+1)$

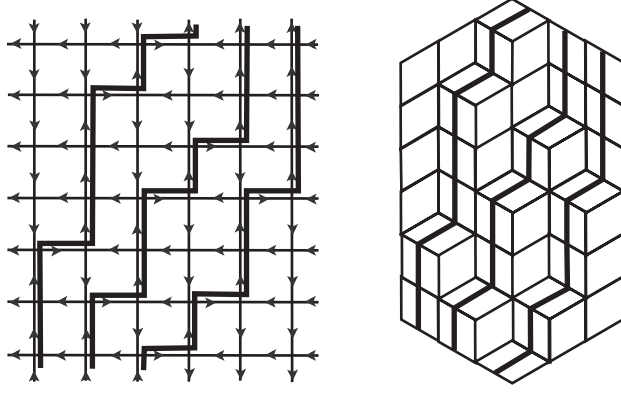


Figure 2: The admissible nest of lattice paths of the four-vertex model and the corresponding strict plane partition.

with the following boundary conditions: all arrows on the left and right boundaries are pointing to the left, while the arrows on the top and bottom of the first N columns (counting from the left) are pointing inwards and the arrows on the top and bottom of the last N ones are pointing outwards (see Fig. 2).

It was proved in [22] that if one puts $v_j = q^{-\frac{j}{2}}$ and $u_j = q^{\frac{j-1}{2}}$ in the scalar product of the state-vectors (49), then

$$\langle \Psi_N(q^{-\frac{1}{2}}, \dots, q^{-\frac{N}{2}}) | \Psi_N(1, \dots, q^{\frac{N-1}{2}}) \rangle = q^{-N^2(N-1)} Z_q^{\text{spp}}(N, N, M), \quad (53)$$

where $Z_q^{\text{spp}}(N, N, M)$ is the generating function of strict plane partitions placed into a box of size $N \times N \times M$:

$$Z_q^{\text{spp}}(N, N, M) \equiv q^{N^2(N-1)} \prod_{1 \leq j, k \leq N} \frac{1 - q^{M+3-j-k}}{1 - q^{j+k-1}}. \quad (54)$$

Being taken at $q = 1$, this formula gives the number of strict plane partitions in a box of size $N \times N \times M$:

$$Z_{q=1}^{\text{spp}}(N, N, M) = \prod_{1 \leq j, k \leq N} \frac{M + 3 - j - k}{j + k - 1}. \quad (55)$$

It is straightforward to obtain [19, 24] that the entry

$$\lim_{q \rightarrow 1} \langle \Psi_N(q^{-\frac{1}{2}}, \dots, q^{-\frac{N}{2}}) | \bar{H}_n | \Psi_N(1, \dots, q^{\frac{N-1}{2}}) \rangle = Z_{q=1}^{\text{spp}}(N, N, M - n - 1) \quad (56)$$

is equal to the number strict partitions in a $N \times N \times (M - n - 1)$ box.

5 Low temperature limit

Let us obtain the low temperature estimate of the correlation function (20). The survival probability of the ferromagnetic string is written as follows:

$$\mathcal{T}(\theta^v, n, \beta) = \frac{|\mathcal{V}(e^{i\theta^v})|^2}{V^2} \sum_{\{\theta\}} e^{-\beta(E_N(\theta) - E_N(\theta^v))} |\mathcal{V}(e^{i\theta}) \mathcal{P}(e^{-i\theta}, e^{i\theta^v})|^2, \quad (57)$$

where $V \equiv (M+1)(M+1-N)^{N-1}$. Here $\mathcal{P}(e^{-i\theta}, e^{i\theta^v})$ is given according to (26), provided the summation domain is given by $\tilde{\Lambda} \subseteq \{(M-2N-n+1)^N\}$.

If the chain is long enough while the number of quasi-particles is moderate, i.e., $N \ll M$, we replace the sums in (57) by the integrals as follows:

$$\begin{aligned} \mathcal{T}(\theta^v, n, \beta) &\simeq \frac{1}{(2\pi)^N N!} \prod_{1 \leq r < s \leq N} 2(1 - \cos(\theta_r^v - \theta_s^v)) \\ &\times \prod_{i=1}^N \left(\int_0^{2\pi} \frac{d\theta_i}{2\pi} \right) e^{\beta \sum_{l=1}^N (\cos \theta_l - \cos \theta_l^v)} |\mathcal{P}(e^{-i\theta}, e^{i\theta^v})|^2 \prod_{1 \leq k < l \leq N} |e^{i\theta_k} - e^{i\theta_l}|^2. \end{aligned} \quad (58)$$

In the considered limit, we may asymptotically put $\cos \theta_l \approx 1$ and $\cos \theta_l^v \approx 1$, $\forall l$. At the large enough β , we approximate (58) in the following way:

$$\mathcal{T}(\theta^v \approx \mathbf{0}, n, \beta) \simeq \frac{\mathcal{I} \mathcal{P}^2(\mathbf{1}, \mathbf{1})}{(2\pi)^N N!} \left(\frac{2}{\beta} \right)^{N^2/2} \prod_{1 \leq r < s \leq N} (r - s)^2, \quad (59)$$

$$\mathcal{I} \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-\sum_{l=1}^N x_l^2} \prod_{1 \leq k < l \leq N} |x_k - x_l|^2 dx_1 dx_2 \dots dx_N. \quad (60)$$

The integral \mathcal{I} (60) is the *Mehta integral*, and its value is known [35]:

$$\mathcal{I} = \frac{\pi^{N/2} \Gamma(1|N+1)}{2^{N(N-1)/2}},$$

where

$$\prod_{1 \leq r < s \leq N} |r - s| = \Gamma(1|N), \quad \Gamma(1|N) \equiv \prod_{n=1}^N \Gamma(n).$$

Eventually, taking into account (56) and (59), we express the answer in the following form:

$$\mathcal{T}_{\text{SA}}(\theta, n, \beta) \simeq (Z_{q=1}^{\text{SPP}}(N, N, M - n - 1))^2 \frac{\mathcal{C}_N}{\beta^{N^2/2}}, \quad \mathcal{C}_N \equiv \frac{\Gamma^3(1|N)}{(2\pi)^{N/2}}.$$

The low temperature decay of the survival probability of the ferromagnetic string is governed by the critical exponent $N^2/2$, while its amplitude is proportional to the squared number of strict plane partitions in a box of size $N \times N \times (M - n - 1)$.

6 Conclusion

In this paper we studied the XXZ Heisenberg magnetic chain in a specific limit of the anisotropy parameter: $\Delta \rightarrow -\infty$. The, so-called, survival probability of the ferromagnetic string has been calculated over the N -particle ground state of the model. It was proved that the Hamiltonian of the Ising limit, previously introduced in the papers [24, 29, 30], is correct as the limit $\Delta \rightarrow -\infty$ of the XXZ Hamiltonian. Finally, we have found the leading term of the low temperature asymptotics of the survival probability of the

ferromagnetic string for the model in the finite volume and at fixed number of quasi-particles. We have demonstrated that the amplitude of the asymptotics is proportional to the squared number of strict plane partitions in a box, while its critical exponent is proportional to the squared number of particles. Comparing this result with that obtained in [24, 37], one can conclude that this type of the behavior is universal for a special class of the integrable models.

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